Journal of the Egyptian Mathematical Society (2015) 23, 278–281

Egyptian Mathematical Society

Journal of the Egyptian Mathematical Society

www.etms-eg.org
www.elsevier.com/locate/joems

ORIGINAL ARTICLE

Exact traveling wave solutions of Kadomtsev–Petviashvili equation

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Received 20 December 2013; revised 2 March 2014; accepted 31 March 2014
Available online 3 May 2014

KEYWORDS

$\exp(-\Phi(\xi))$-expansion method;
KP equation;
Exact solution;
NLEEs

Abstract In this paper, the $\exp(-\Phi(\xi))$-expansion method with the aid of Maple has been used to obtain the exact solutions of the Kadomtsev–Petviashvili (KP) equation. Each of the obtained solutions, namely hyperbolic function solutions, trigonometric function solutions and rational function solutions, contain an explicit function of the variables in the considered equation. It has been shown that the method provides a powerful mathematical tool for solving nonlinear wave equations in mathematical physics and engineering problems.

2010 MATHEMATICS SUBJECT CLASSIFICATION: 35K99; 35P05; 35P99

1. Introduction

Nonlinear evolution equations (NLEEs) i.e., partial differential equations with time t as one of the independent variables have become a useful tool for describing the natural phenomena of science and engineering. NLEEs arise not only from many fields of mathematics, but also from other branches of science such as physics, mechanics and material science. Exact solutions of NLEEs play an important role in the proper understanding of qualitative features of many phenomena and processes in various areas of natural science. Even those special exact solutions that do not have a clear physical meaning can be used as test problems to verify the consistency and estimate errors of various numerical, asymptotic, and approximate analytical methods. Exact solutions can serve as a basis for perfecting and testing computer algebra software packages for solving NLEEs. It is significant that many equations of physics, chemistry, and biology contain empirical parameters or empirical functions. Exact solutions allow researchers to design and run experiments, by creating appropriate natural conditions, to determine these parameters or functions. Therefore, investigation of exact traveling wave solutions is becoming successively attractive in nonlinear sciences day by day. However, not all equations posed of these models are solvable. Hence it becomes increasingly important to be familiar with all traditional and recently developed methods for solving these models and the implementation of new methods. As a result, many new techniques have been successfully developed by diverse groups of mathematicians and physicists, such as, the Kudryashov method [1], the Exp-function method [2–5], the

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Peer review under responsibility of Egyptian Mathematical Society.

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http://dx.doi.org/10.1016/j.joems.2014.03.010

Homotopy perturbation method [6,7], the modified simple equation method [8–10], the (G'/G)-expansion method [11–15], the $\exp(-\Phi(\xi))$-expansion method [16].

The objective of this article is to apply the $\exp(-\Phi(\xi))$-expansion method to construct the exact solutions for nonlinear evolution equations in mathematical physics via the KP equation.

The article is prepared as follows: In Section 2, the $\exp(-\Phi(\xi))$-expansion method is discussed; In Section 3, we apply this method to the nonlinear evolution equation pointed out above and in Section 4 conclusions are given.

2. Algorithm of the exp(−Φ(ξ))-expansion method

In this section we will describe the algorithm of the $\exp(-\Phi(\xi))$-expansion method for finding traveling wave solutions of nonlinear evolution equations. Suppose that a nonlinear equation, say in two independent variables x and t is given by

$$P(u, u_t, u_x, u_{tt}, u_{xx}, u_{xt}, \ldots\ldots\ldots\ldots) = 0, \tag{2.1}$$

where $u(\xi) = u(x,t)$ is an unknown function, P is a polynomial of $u(x,t)$ and its partial derivatives in which the highest order derivatives and nonlinear terms are involved. In the following, we give the main steps of this method [16]:

Step 1. Combining the independent variables x and t into one variable $\xi = x \pm \omega t$, we suppose that

$$u(x,t) = u(\xi), \quad \xi = x \pm \omega t, \tag{2.2}$$

The traveling wave transformation Eq. (2.2) permits us to reduce Eq. (2.1) to the following ordinary differential equation (ODE):

$$Q(u, u', u'', \ldots\ldots\ldots) = 0. \tag{2.3}$$

where Q is a polynomial in $u(\xi)$ and its derivatives, whereas $u'(\xi) = \frac{du}{d\xi}$, $u''(\xi) = \frac{d^2u}{d\xi^2}$, and so on.

Step 2. We suppose that Eq. (2.3) has the formal solution

$$u(\xi) = \sum_{i=0}^{n} \alpha_i (\exp(-\Phi(\xi)))^i, \tag{2.4}$$

where α_i $(0 \leqslant i \leqslant n)$ are constants to be determined, such that $\alpha_n \neq 0$, and $\Phi = \Phi(\xi)$ satisfies the following ODE:

$$\Phi'(\xi) = \exp(-\Phi(\xi)) + \mu \exp(\Phi(\xi)) + \lambda, \tag{2.5}$$

Eq. (2.5) gives the following solutions: When $\lambda^2 - 4\mu > 0$, $\mu \neq 0$,

$$\Phi(\xi) = \ln\left(\frac{-\sqrt{(\lambda^2-4\mu)}\tanh\left(\frac{\sqrt{(\lambda^2-4\mu)}}{2}(\xi+k)\right) - \lambda}{2\mu}\right), \tag{2.6}$$

$$\Phi(\xi) = \ln\left(\frac{-\sqrt{(\lambda^2-4\mu)}\coth\left(\frac{\sqrt{(\lambda^2-4\mu)}}{2}(\xi+k)\right) - \lambda}{2\mu}\right), \tag{2.7}$$

When $\lambda^2 - 4\mu < 0$, $\mu \neq 0$,

$$\Phi(\xi) = \ln\left(\frac{\sqrt{(4\mu-\lambda^2)}\tan\left(\frac{\sqrt{(4\mu-\lambda^2)}}{2}(\xi+k)\right) - \lambda}{2\mu}\right), \tag{2.8}$$

$$\Phi(\xi) = \ln\left(\frac{\sqrt{(4\mu-\lambda^2)}\cot\left(\frac{\sqrt{(4\mu-\lambda^2)}}{2}(\xi+k)\right) - \lambda}{2\mu}\right), \tag{2.9}$$

When $\lambda^2 - 4\mu > 0$, $\mu = 0$, $\lambda \neq 0$,

$$\Phi(\xi) = -\ln\left(\frac{\lambda}{\exp(\lambda(\xi+k)) - 1}\right), \tag{2.10}$$

When $\lambda^2 - 4\mu = 0$, $\mu \neq 0$, $\lambda \neq 0$,

$$\Phi(\xi) = \ln\left(-\frac{2(\lambda(\xi+k)+2)}{\lambda^2(\xi+k)}\right), \tag{2.11}$$

When $\lambda^2 - 4\mu = 0$, $\mu = 0$, $\lambda = 0$,

$$\Phi(\xi) = \ln(\xi + k), \tag{2.12}$$

where k is an arbitrary constant and $\alpha_n, \ldots\ldots, \omega, \lambda, \mu$ are constants to be determined later, $\alpha_n \neq 0$, the positive integer n can be determined by considering the homogeneous balance between the highest order derivatives and the nonlinear terms appearing in Eq. (2.3).

Step 3. We substitute Eq. (2.4) into (2.3) and then we account the function $\exp(-\Phi(\xi))$. As a result of this substitution, we get a polynomial of $\exp(-\Phi(\xi))$. We equate all the coefficients of same power of $\exp(-\Phi(\xi))$ to zero. This procedure yields a system of algebraic equations whichever can be solved to find $\alpha_n, \ldots\ldots, \omega, \lambda, \mu$. Substituting the values of $\alpha_n, \ldots\ldots, \omega, \lambda, \mu$ into Eq. (2.4) along with general solutions of Eq. (2.5) completes the determination of the solution of Eq. (2.1).

3. Application

The Kadomtsev–Petviashvili (KP) equation

$$u_{xt} - 6uu_{xx} - 6(u_x)^2 + u_{xxxx} + 3\delta^2 u_{yy} = 0, \tag{3.1}$$

or $(u_t - 6uu_x + u_{xxx})_x + 3\delta^2 u_{yy} = 0$,

is a two-dimensional generalization of the KdV equation. Kadomtsev and Petviashili (1970) first introduced this equation to describe slowly varying nonlinear waves in a dispersive medium [17,18]. Eq. (3.1) with $\delta^2 = +1$ arises in the study of weakly nonlinear dispersive waves in plasmas and also in the modulation of weakly nonlinear long water waves [19] which travel nearly in one dimension (that is, nearly in a vertical plane). The equation with $\delta^2 = -1$ arises in acoustics and admits unstable soliton solutions, whereas for $\delta^2 = +1$ the solitons are stable.

The traveling wave transformation equation is

$$u = u(x,y,t), \quad \xi = x + y - \omega t, \quad u = u(\xi), \quad u(x,y,t) = u(\xi). \tag{3.2}$$

Using traveling wave Eq. (3.2), Eq. (3.1) reduces into the following ODE

$$(-\omega u' - 6uu' + u''')' + 3\delta^2 u'' = 0. \tag{3.3}$$

Integrating Eq. (3.3) twice with respect to ξ setting constant of integration to zero, we obtain the following ODE

$$u'' + (3\delta^2 - \omega)u - 3u^2 = 0. \tag{3.4}$$

Now balancing the highest order derivative u'' and non-linear term u^2, we get $n = 2$.

Now for $n = 2$,

$$u(\xi) = \alpha_0 + \alpha_1 \exp(-\Phi(\xi)) + \alpha_2(\exp(-\Phi(\xi)))^2. \tag{3.5}$$

where α_0, α_1, and α_2 are constants to be determined such that $\alpha_2 \neq 0$, while λ, μ are arbitrary constants.

Substituting Eq. (3.5) into Eq. (3.4) and then equating the coefficients of $\exp(-\Phi(\xi))^j$ to zero, where $j \geqslant 0$, we get

$$6\alpha_2\mu\lambda - \omega\alpha_1 + \alpha_1\lambda^2 + 2\alpha_1\mu - 6\alpha_0\alpha_1 + 3\sigma^2\alpha_1 = 0. \tag{3.6}$$

$$-6\alpha_0\alpha_2 + 3\sigma^2\alpha_2 + 3\alpha_1\lambda + 4\alpha_2\lambda^2 - 3\alpha_1^2 + 8\alpha_2\mu - \omega\alpha_2 = 0. \tag{3.7}$$

$$10\alpha_2\lambda - 6\alpha_1\alpha_2 + 2\alpha_1 = 0. \tag{3.8}$$

$$-3\alpha_2^2 + 6\alpha_2 = 0. \tag{3.9}$$

$$2\alpha_2\mu^2 + \alpha_1\mu\lambda + 3\sigma^2\alpha_0 - \omega\alpha_0 - 3\alpha_0^2 = 0. \tag{3.10}$$

Solving Eqs. (3.6)–(3.10) by using Maple 13, we find that solution of Eq. (3.1) exists only in the following two cases:

Set 1: $\omega = 3\delta^2 - 4\mu + \lambda^2$, $\alpha_0 = 2\mu$, $\alpha_1 = 2\lambda$, $\alpha_2 = 2$.
Set 2: $\omega = 4\mu - \lambda^2 + 3\delta^2$, $\alpha_0 = \frac{2}{3}\mu + \frac{1}{3}\lambda^2$, $\alpha_1 = 2\lambda$, $\alpha_2 = 2$.

where μ and λ are arbitrary constants.

Case 1: Solution of Eq. (3.1) which corresponds to Set 1 is given by

$$u(\xi) = 2\mu + 2\lambda\exp(-\Phi(\xi)) + 2(\exp(-\Phi(\xi)))^2. \tag{3.11}$$

where $\xi = x + y - (3\delta^2 - 4\mu + \lambda^2)t$.

Now substituting Eqs. (2.6)–(2.12) into Eq. (3.11) respectively, we get the following seven traveling wave solutions of KP equation.

When $\lambda^2 - 4\mu > 0$, $\mu \neq 0$,

$$u_1(\xi) = 2\mu - \frac{4\lambda\mu}{\sqrt{(\lambda^2-4\mu)}\tanh\left(\frac{1}{2}\sqrt{(\lambda^2-4\mu)}(\xi+k)\right)+\lambda} + \frac{8\mu^2}{\left(\sqrt{(\lambda^2-4\mu)}\tanh\left(\frac{1}{2}\sqrt{(\lambda^2-4\mu)}(\xi+k)\right)+\lambda\right)^2},$$

$$u_2(\xi) = 2\mu - \frac{4\lambda\mu}{\sqrt{(\lambda^2-4\mu)}\coth\left(\frac{1}{2}\sqrt{(\lambda^2-4\mu)}(\xi+k)\right)+\lambda} + \frac{8\mu^2}{\left(\sqrt{(\lambda^2-4\mu)}\coth\left(\frac{1}{2}\sqrt{(\lambda^2-4\mu)}(\xi+k)\right)+\lambda\right)^2},$$

where $\xi = x - (3\delta^2 - 4\mu + \lambda)t$, k is an arbitrary constant.

When $\lambda^2 - 4\mu < 0$, $\mu \neq 0$,

$$u_3(\xi) = 2\mu + \frac{4\lambda\mu}{\sqrt{(4\mu-\lambda^2)}\tan\left(\frac{1}{2}\sqrt{(4\mu-\lambda^2)}(\xi+k)\right)-\lambda} + \frac{8\mu^2}{\left(\sqrt{(4\mu-\lambda^2)}\tan\left(\frac{1}{2}\sqrt{(4\mu-\lambda^2)}(\xi+k)\right)-\lambda\right)^2},$$

$$u_4(\xi) = 2\mu + \frac{4\lambda\mu}{\sqrt{(4\mu-\lambda^2)}\cot\left(\frac{1}{2}\sqrt{(4\mu-\lambda^2)}(\xi+k)\right)-\lambda} + \frac{8\mu^2}{\left(\sqrt{(4\mu-\lambda^2)}\cot\left(\frac{1}{2}\sqrt{(4\mu-\lambda^2)}(\xi+k)\right)-\lambda\right)^2},$$

where $\xi = x - (3\delta^2 - 4\mu + \lambda)t$, k is an arbitrary constant.

When $\lambda^2 - 4\mu > 0$, $\mu = 0$, $\lambda \neq 0$,

$$u_5(\xi) = \frac{2\lambda^2\exp(\lambda(\xi+k))}{(\exp(\lambda(\xi+k))-1)^2},$$

where $\xi = x - (3\delta^2 + \lambda)t$, k is an arbitrary constant.

When $\lambda^2 - 4\mu = 0$, $\mu \neq 0$, $\lambda \neq 0$,

$$u_6(\xi) = \frac{2\lambda^2}{(\lambda(\xi+k)+2)^2},$$

where $\xi = x - (3\delta^2 - 4\mu + \lambda)t$, k is an arbitrary constant.

When $\lambda^2 - 4\mu = 0$, $\mu = 0$, $\lambda = 0$,

$$u_7(\xi) = \frac{2}{(\xi+k)^2},$$

where $\xi = x - 3\delta^2 t$, k is an arbitrary constant.

Case 2: Solution of Eq. (3.1) which corresponds to Set 2 is given by

$$u(\xi) = \frac{2}{3}\mu + \frac{1}{3}\lambda^2 + 2\lambda\exp(-\Phi(\xi)) + 2(\exp(-\Phi(\xi)))^2. \tag{3.12}$$

where $\xi = x + y - (4\mu - \lambda^2 + 3\delta^2)t$.

Now substituting Eqs. (2.6)–(2.12) into Eq. (3.12) respectively, we get the following seven traveling wave solutions of KP equation.

When $\lambda^2 - 4\mu > 0$, $\mu \neq 0$,

$$u_8(\xi) = \frac{2}{3}\mu + \frac{1}{3}\lambda^2 - \frac{4\lambda\mu}{\sqrt{(\lambda^2-4\mu)}\tanh\left(\frac{1}{2}\sqrt{(\lambda^2-4\mu)}(\xi+k)\right)+\lambda} + \frac{8\mu^2}{\left(\sqrt{(\lambda^2-4\mu)}\tanh\left(\frac{1}{2}\sqrt{(\lambda^2-4\mu)}(\xi+k)\right)-\lambda\right)^2},$$

$$u_9(\xi) = \frac{2}{3}\mu + \frac{1}{3}\lambda^2 - \frac{4\lambda\mu}{\sqrt{(\lambda^2-4\mu)}\coth\left(\frac{1}{2}\sqrt{(\lambda^2-4\mu)}(\xi+k)\right)+\lambda} + \frac{8\mu^2}{\left(\sqrt{(\lambda^2-4\mu)}\coth\left(\frac{1}{2}\sqrt{(\lambda^2-4\mu)}(\xi+k)\right)-\lambda\right)^2},$$

where $\xi = x + y - (4\mu - \lambda^2 + 3\delta^2)t$, k is an arbitrary constant.

When $\lambda^2 - 4\mu < 0$, $\mu \neq 0$,

$$u_{10}(\xi)=\frac{2}{3}\mu+\frac{1}{3}\lambda^2+\frac{4\lambda\mu}{\sqrt{(4\mu-\lambda^2)}\tan\left(\frac{1}{2}\sqrt{(4\mu-\lambda^2)}(\xi+k)\right)-\lambda}+\frac{8\mu^2}{\left(\sqrt{(4\mu-\lambda^2)}\tan\left(\frac{1}{2}\sqrt{(4\mu-\lambda^2)}(\xi+k)\right)-\lambda\right)^2},$$

$$u_{11}(\xi)=\frac{2}{3}\mu+\frac{1}{3}\lambda^2+\frac{4\lambda\mu}{\sqrt{(4\mu-\lambda^2)}\cot\left(\frac{1}{2}\sqrt{(4\mu-\lambda^2)}(\xi+k)\right)-\lambda}+\frac{8\mu^2}{\left(\sqrt{(4\mu-\lambda^2)}\cot\left(\frac{1}{2}\sqrt{(4\mu-\lambda^2)}(\xi+k)\right)-\lambda\right)^2},$$

where $\xi = x + y - (4\mu - \lambda^2 + 3\delta^2)t$, k is an arbitrary constant.

When $\lambda^2 - 4\mu > 0$, $\mu = 0$, $\lambda \neq 0$,

$$u_{12}(\xi) = \frac{1}{3}\lambda^2 + \frac{2\lambda^2}{\exp(\lambda(\xi+k)-1)} + \frac{2\lambda^2}{(\exp(\lambda(\xi+k)-1))^2},$$

where $\xi = x + y - (3\delta^2 - \lambda^2)t$, k is an arbitrary constant.

When $\lambda^2 - 4\mu = 0$, $\mu \neq 0$, $\lambda \neq 0$,

$$u_{13}(\xi) = \frac{2\lambda^2}{(\lambda(\xi+k)+2)^2},$$

where $\xi = x + y - (4\mu - \lambda^2 + 3\delta^2)t$, k is an arbitrary constant.

When $\lambda^2 - 4\mu = 0$, $\mu = 0$, $\lambda = 0$,

$$u_{14}(\xi) = \frac{2}{(\xi+k)^2},$$

where $\xi = x + y - 3\delta^2 t$, k is an arbitrary constant.

Remark. With the aid of Maple 13, we have assured the correctness of the obtained solutions by putting them back into the original equation.

4. Conclusions

This study shows that the $\exp(-\Phi(\xi))$-expansion method is quite efficient and practically well suited for use in finding new traveling wave solutions for the Kadomtsev–Petviashvili equation. The reliability of the method and the reduction in the size of computational domain give this method a wider applicability. We obtain some new traveling wave solutions including hyperbolic function solutions, trigonometric function solutions and rational solutions. The results show that the method is reliable and effective and gives more solutions. We hope that the obtained results will be useful for further studies in mathematical physics and engineering.

Acknowledgements

The authors would like to express thanks to the anonymous referees for their useful and valuable comments and suggestions.

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